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High orders of Weyl series: resurgence for odd balls

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Abstract. The semiclassical Weyl series for the d-dimension, unit-radius sphere quantum billiard is studied. A conjecture of Berry and Howls (1994 *Proc. R. Soc.* **447** 527–55) on the late terms of such series for two-dimensional billiards is seen to survive for general (integer) dimension. The conjecture postulates a leading-order, factorial-on-power approximation for the late terms of the Weyl series in terms of the length of a periodic orbit of the classical system. The expansions manifest a difference between odd and even dimensions. The dominating orbit is the diametral, length-4 path in the even-dimension spheres, echoing the known result for the circle billiard. However, when d is odd, it is the next-longest orbit. This surprise can be traced to an 'accidental' symmetry in a postulated hyperasymptotic remainder term. Higher-order asymptotic correction terms are found confirming the resurgent link of the Weyl series to the low orders of the oscillatory periodic orbit corrections. From the structure of the latter, it is possible to make further conjectures on the late terms of the periodic orbit corrections themselves. A factorial-on-power behaviour is also found, but now involving the differences between p-bounce orbits and associated whispering-gallery modes.

1. Introduction

Over the past three decades, the asymptotic expansions of smoothed spectral functions associated with Schrödinger equations within d-dimensional balls have received aperiodic attention (Stewartson and Waechter 1971, Waechter 1972, Kennedy 1978, 1979, Berry and Howls 1994, Bordag *et al* 1996a, b, c, Levitin 1998). Recently Bordag *et al* (1996a, b, c) have described an analytic technique which combines the explicit form of the known eigenfunctions with contour integral techniques to generate high-energy expansions of the d-balls' spectral ζ-functions. For d-dimensional balls the eigenfunctions are simply Bessel functions, and hence Dirichlet, Neumann and Robin boundary data have all been dealt with (e.g., Dowker 1996). The limit on the number of terms which can be derived is now only the size of the computer memory available. Dowker *et al* (1996) and Elizalde *et al* (1993) have performed similar calculations for spinors and other equations within ball domains. Levitin (1998) has provided relations which generate coefficients of spectral expansions for balls in several dimensions simultaneously.

Beyond calculating the terms in the asymptotics Weyl series, an understanding of their structure seems desirable. Berry and Howls (1994, hereafter called BH) considered the late terms in the high-energy expansion of the regularized resolvent within a circular domain with Dirichlet boundaries

$$g(s) \equiv \lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{E_n+s^2}-\frac{A}{4\pi}\ln\left(\frac{E_N}{s^2}\right)\right]. \tag{1}$$

The resolvent has been defined here in terms of a complex energy variable $s = -\mathrm{i}\sqrt{E}$ so that corrections arising from periodic orbits l_j are exponentially small in the region where the Weyl series is initially analysed. In a sector surrounding the Im $(s) > 0$ axis, the large s expansion is

$$g(s) \sim \sum_{r=1}^{\infty} \frac{c_r}{s^r} + \sum_j \mathrm{e}^{-sl_j} \sum_{r=1}^{\infty} \frac{c_r^{(j)}}{s^r}. \tag{2}$$

BH conjectured that the c_r behaved asymptotically as

$$c_r \sim \frac{\alpha(r+\beta)!}{l^r} \tag{3}$$

where l is an *associated* (not necessarily *actual*) periodic orbit of the billiard, β a constant determined by the character of the orbit and α is intimately related to the first term of the oscillatory periodic orbit correction pertaining to l, e.g., possibly one of the $c_0^{(j)}$. They provided numerical evidence for the result, via a Borel sum of the conjectured later terms. Howls and Trasler (1998, hereafter called HT) extended the conjecture in the case of 2D cake-slice orbits with Dirichlet conditions demonstrating analytically and numerically a higher-order version of the asymptotic form

$$c_r \sim \sum_j \sum_{k=0}^{\infty} \frac{\alpha_k^{(j)}(r+\beta_j-k)!}{l_j^{r-k}} \tag{4}$$

where the $\{l_j\}$ are all associated periodic orbits. (Note that the index j is for identification purposes only and does not necessarily label the number of bounces the orbit l_j makes on the boundary.) In this paper we analyse the late terms of d-balls with Dirichlet boundary data (other conditions will be considered elsewhere). The results bear out the extended conjecture of HT, but with an important qualification. It is known that the shortest orbit may not dominate the Weyl series in *concave* billiards (BH) or in billiards with corners (HT). However, we shall demonstrate that this phenomenon can also occur in *convex* C^{∞} cavities, specifically odd-dimensional balls. The reasons underlying this surprise are explained.

In section 2 we outline the algorithm to generate the coefficients in any (integer) dimension. In section 3 the leading-order behaviour and corrections to the late terms behaviour are explored numerically. In section 4 these estimates are justified analytically. The resurgence relationship between the c_r and that of the periodic orbit coefficients $(c_r^{(j)})$ is highlighted. In section 5 we explain the absence of influence of the shortest orbit on the odd-dimensional Weyl terms. The results of the first part of the paper allow us to make conjectures about high orders of the periodic orbit corrections $c_r^{(j)}$ in section 6. We conclude with a discussion in section 7.

2. Algorithm for generating d-ball coefficients

Many authors have considered the problem of generating coefficients in the Weyl expansion for d-dimensional balls. Stewartson and Waechter (1971) provided a method for Dirichlet conditions in $d = 2$. Waechter (1972) extended this to $d = 3$, but neglected an important exponential contribution which was picked up by Kennedy (1979). Bordag *et al* (1996a, b, c) provided an alternative general method for generating the coefficients for billiards subject to general boundary data, provided however that the exact eigenfunction expansions are known. This integral technique automatically accounts for Waechter's overlooked exponentials. Levitin's method (1998) is based on relations between balls of different dimensions and claims computational efficiency.

Here we will follow the approach of Kennedy (1979). Apart from defining the relevant quantities, the inclusion of this calculation can be justified for several reasons. First, even the most recent publications (e.g., Brack and Bhaduri 1997) propagate Waechter's (1972) mistake. Secondly, the Kennedy formalism currently lends itself more immediately to the resurgence of the periodic orbit contributions in the late terms c_r. Thirdly, because the inclusion of oscillatory terms to generate algebraic coefficients is a salutary lesson for semiclassical expanders, especially as we are dealing here with C^∞ boundaries. The method for generating $d = 2$ Weyl coefficients with Dirichlet boundary data has been outlined in BH.

In general-d, the regularized resolvent for a quantum billiard Ω is defined in terms of the full and free Green functions G and G_0, respectively, as

$$g(s) = \int_\Omega \mathrm{d}\boldsymbol{r} \lim_{\boldsymbol{r}_0\to\boldsymbol{r}} [G(\boldsymbol{r}, \boldsymbol{r}_0; s) - G_0(\boldsymbol{r}, \boldsymbol{r}_0; s)]. \tag{5}$$

Thus we need to solve

$$(-\nabla^2 + s^2)G(\boldsymbol{r}, \boldsymbol{r}_0; s) = \delta(\boldsymbol{r} - \boldsymbol{r}_0) \qquad \boldsymbol{r}, \boldsymbol{r}_0 \in \Omega \tag{6}$$

where Ω is a d-dimensional ball of unit radius. The Green function is decomposed as

$$G(\boldsymbol{r}, \boldsymbol{r}_0; s) = G_0(\boldsymbol{r}, \boldsymbol{r}_0; s) + \chi(\boldsymbol{r}, \boldsymbol{r}_0; s) \tag{7}$$

where χ is the compensating function appropriate to the boundary data.

We shall find it convenient to define

$$\nu = \frac{d}{2} - 1 \tag{8}$$

so that ν is (half-) integer in (odd) even dimensions. In arbitrary-d, the free Green function can be written in spherical geometries as (Balian and Bloch 1972)

$$G_0(\boldsymbol{r}, \boldsymbol{r}_0; s) = \frac{s^{2\nu}}{(2\pi)^{\nu+1}(sR)^\nu} K_\nu(sR) \qquad R = |\boldsymbol{r} - \boldsymbol{r}_0| \tag{9}$$

which for $d \geqslant 3$ may be expanded thus:

$$\frac{K_\nu(sR)}{(sR)^\nu} = 2^\nu(\nu-1)! \sum_{m=0}^{\infty}(m+\nu)\frac{I_{m+\nu}(sr_<)}{(sr_<)^\nu}\frac{K_{m+\nu}(sr_>)}{(sr_>)^\nu} C_m^{(\nu)}(\cos\theta) \tag{10}$$

where θ is the angle subtended by the vector $\boldsymbol{r} - \boldsymbol{r}_0$ at the origin, $r = |\boldsymbol{r}|$, $r_0 = |\boldsymbol{r}_0|$, $r_< = \min(r, r_0)$ and $r_> = \max(r, r_0)$. The presence of the Gegenbauer polynomials $C_m^{(\nu)}$ naturally accounts for the eigenvalue degeneracies as $\boldsymbol{r}_0 \to \boldsymbol{r}$ in (5).

2.1. *Coefficients for $d \geqslant 3$, Dirichlet data*

Using the expansion (10) for the free Green function, when Ω is a unit d-ball the boundary data is satisfied by the compensating function

$$\chi = -\frac{(\nu-1)!}{2\pi^{\nu+1}} \sum_{m=0}^{\infty}(m+\nu)\frac{I_{m+\nu}(s)}{K_{m+\nu}(s)}\frac{I_{m+\nu}(sr_<)I_{m+\nu}(sr_>)}{r_<^\nu r_>^\nu} C_m^{(\nu)}(\cos\theta).$$

Thus, on taking the limits in the resolvent:

$$\lim_{\boldsymbol{r}_0\to\boldsymbol{r}} \chi = -\frac{(\nu-1)!}{2\pi^{\nu+1}} \sum_{m=0}^{\infty}(m+\nu)\frac{I_{m+\nu}(s)}{K_{m+\nu}(s)}\frac{I_{m+\nu}^2(sr)}{r^{2\nu}} C_m^{(\nu)}(1) \tag{11}$$

and so we obtain an expression for the resolvent

$$g(s) = -\frac{(\nu-1)!}{2\pi^{\nu+1}} \sum_{m=0}^{\infty} \frac{(m+\nu)(m+2\nu-1)!}{m!(2\nu-1)!}\frac{K_{m+\nu}(s)}{I_{m+\nu}(s)} \int_\Omega \mathrm{d}V\, \frac{I_{m+\nu}^2(sr)}{r^{2\nu}}. \tag{12}$$

Using the relevant spherical polar Jacobian and performing the angular integrations we arrive at

$$g(s) = -\frac{1}{\nu}\sum_{m=0}^{\infty}\frac{(m+\nu)(m+2\nu-1)!}{m!(2\nu-1)!}\frac{K_{m+\nu}(s)}{I_{m+\nu}(s)}\int_0^1 \mathrm{d}r\, r I_{m+\nu}^2(sr) \tag{13}$$

$$= -\sum_{m=0}^{\infty}\frac{(m+\nu)(m+2\nu-1)!}{m!(2\nu)!} f_{m+\nu}(s) \tag{14}$$

where the f_m is defined as

$$f_m(s) = \left(1+\frac{m^2}{s^2}\right) I_m(s)K_m(s) - I_m'(s)K_m'(s) - \frac{I_m'(s)}{sI_m(s)}. \tag{15}$$

Note the similarity with the result (54) of BH for $d = 2$, but with the important difference in the range of summation.

We now sum the series in (13) by using the half-range Poisson sum formula

$$\sum_{m=0}^{\infty} h_{m+\nu}(s) = \sum_{\mu=-\infty}^{\infty}(-)^{2\mu\nu}\int_0^{\infty}\mathrm{d}m\, h_m(s)\mathrm{e}^{2\pi \mathrm{i} m\mu} \tag{16}$$

generating the integral expression

$$g(s) = -\frac{1}{(2\nu)!}\sum_{\mu=-\infty}^{\infty}(-)^{2\mu\nu}\int_0^{\infty}\mathrm{d}m\,\frac{m(m+\nu-1)!}{(m-\nu)!} f_m(s)\mathrm{e}^{2\pi \mathrm{i} m\mu}. \tag{17}$$

The factorials simplify when d is integer revealing the appropriate degeneracies of the eigenvalues and the integrands become

$$m f_m(s)\mathrm{e}^{2\pi \mathrm{i} m\mu}\prod_{n=1}^{\nu-1/2}(m^2-(n-\tfrac{1}{2})^2) \qquad d \text{ odd} \tag{18}$$

$$m^2 f_m(s)\mathrm{e}^{2\pi \mathrm{i} m\mu}\prod_{n=1}^{\nu-1}(m^2-n^2) \qquad d \text{ even.} \tag{19}$$

From these results it is clear that the d-ball's resolvent can be expressed in terms of the $(d-2)$-, $(d-4)$-, ... balls', via an expansion of the products in (18), (19). Without loss of generality we can now focus on the integrand with the highest power of m:

$$L_\nu(s) = -\frac{1}{(2\nu)!}\sum_{\mu=-\infty}^{\infty}(-)^{2\mu\nu}\int_0^{\infty}\mathrm{d}m\, m^{2\nu} f_m(s)\mathrm{e}^{2\pi \mathrm{i} m\mu}. \tag{20}$$

Denoting by W_d the Weyl series for the d-ball, we can determine odd-dimensional d-Weyl series as

$$\begin{aligned} W_6 &= L_2(s) - \tfrac{1}{12}W_4 \\ W_8 &= L_3(s) - \tfrac{1}{360}W_4 - \tfrac{1}{6}W_6 \end{aligned} \tag{21}$$

while that for the even case is

$$\begin{aligned} W_5 &= L_{3/2}(s) - \tfrac{1}{24}W_3 \\ W_7 &= L_{5/2}(s) - \tfrac{1}{1920}W_3 - \tfrac{1}{8}W_5 \\ W_9 &= L_{7/2}(s) - \tfrac{1}{322\,560}W_3 - \tfrac{13}{1920}W_5 - \tfrac{5}{24}W_7 \end{aligned} \tag{22}$$

and so on.

Thus there are only two analytically different base cases, namely $L_{1/2}$ and L_1, all higher-dimensional coefficients being generated from these. In recognizing (20)–(22) we see that, assuming that the lower-dimensional coefficients have been generated, the additional effort required to calculate in a higher dimension is constant. Moreover, because of the analytic bases W_3 and W_4, it will transpire that it is possible to make detailed asymptotic statements about the form of coefficients in any dimension (cf Levitin 1998).

It is now necessary to split our treatment into consideration of even and odd dimensions. To that end we separate the large-s expansion of (20) into two parts, and consider the c_r to be defined as

$$c_r = z_r + h_r \tag{23}$$

where $\mu = 0$ and $\mu \neq 0$ respectively.

2.2. *The zeroth harmonic*

Setting $\mu = 0$ in (20) we expand the resulting integral using the Debye expansions of the Bessel functions (BH 55) and changing the variable $m = xs$ we observe that

$$z_r = \frac{1}{(d-2)!}\int_0^\infty \mathrm{d}x\, \frac{\sqrt{1+x^2}}{x^r} B_{r+d-2}\left(\frac{x}{\sqrt{1+x^2}}\right) \tag{24}$$

where the $\{B_r\}$ are given by BH (58). The integrals are straightforward to evaluate: each integrand reduces to the form (BH 57)

$$\sum_{k=0}^{r+2\nu-1} \frac{q_{r,2k+2\nu} x^{2k+2\nu}}{(1+x^2)^{3r/2+3\nu-1/2}} \tag{25}$$

for constants $q_{r,2k+2\nu}$ and are thus effectively the same as those in BH. Note that the $I_m K_m$ and $I'_m K'_m$ coefficients do not contribute beyond c_2, thus simplifying the computational effort required (HT). The calculation (24) is common to both even and odd dimensions.

2.3. *The higher harmonics*

The $\mu \neq 0$ calculation can be written in terms of two Fourier transforms.

$$L_\nu^{(\mu\neq 0)}(s) = -\frac{1}{(2\nu)!}\sum_{\mu=1}^{\infty}(-)^{2\mu\nu}\int_0^\infty \mathrm{d}m\, m^{2\nu} f_m(s)(\mathrm{e}^{2\pi \mathrm{i} m\mu} + \mathrm{e}^{-2\pi \mathrm{i} m\mu}). \tag{26}$$

For the large-s expansion we use the asymptotics of Fourier integrals with continuous derivatives on the real axis (Olver 1997)

$$\int_0^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{i}\sigma x} f(x) \sim \sum_{n=1}^{\infty}\left(\frac{\mathrm{i}}{\sigma}\right)^n f^{(n-1)}(0) \qquad s \gg 1. \tag{27}$$

Consideration of the form of the integrands (25) shows that after appropriate cancellations (26) can be expanded as

$$L_\nu^{(\mu\neq 0)}(s) \sim \frac{1}{(2\nu)!}\sum_{r=1-2\nu}^{\infty}\sum_{k=0}^{r+2\nu-1}\sum_{n=1}^{\infty}\frac{q_{r,2k+2\nu}}{s^{r+2n}}(\partial_x^{2n-1}X_{k,\nu}^{(r)})_{x=0}\frac{2(-)^{n+1}}{(2\pi)^{2n}}Z(2n) \tag{28}$$

where

$$X_{k,\nu}^{(r)} = \frac{x^{2(\kappa+\nu)}}{(1+x^2)^{\frac{3r}{2}+3\nu+\frac{1}{2}}} \tag{29}$$

and (Abramowitz and Stegun 1972 section 23.2)

$$Z(2n) = \zeta(2n) \times \begin{cases} (1-2^{1-2n}) & \nu \text{ integer-plus-half} \\ 1 & \nu \text{ integer} \end{cases} \tag{30}$$

in terms of the Riemann ζ-function. Since $X_{k,\nu}^{(r)}$ is even for $2\nu =$ even, i.e. $d =$ even, clearly its odd derivatives at the origin are zero. Thus there are no contributions to the algebraic Weyl series from higher harmonic in even-d balls.

In odd-d balls the odd derivatives of the odd $X_{k,\nu}^{(r)}$ are non-zero. Thus the oscillatory higher harmonics may not be neglected and generate an algebraic contribution to the Weyl series. A short but messy calculation gives

$$L_\nu^{(\mu\neq 0)}(s) \sim \sum_{r=2}^{\infty} \frac{h_r}{s^r}$$
$$h_r = \frac{-2}{(2\nu)!} \sum_{k=0}^{\lfloor (r-2)/3 \rfloor} \sum_{n=k+\nu+\frac{1}{2}}^{\lfloor (r-k)/2 \rfloor + \nu - \frac{1}{2}} (-)^{k+\nu+\frac{1}{2}} q_{r-2n,2k+2\nu} \times \frac{(3r/2-2n+2\nu-k-2)!(2n-1)!}{(3r/2-3n+3\nu-3/2)!(n-k-\nu-1/2)!} \frac{Z(2n)}{(2\pi)^{2n}} \tag{31}$$

for d odd ($h_r = 0$ for d even).

Waechter's (1972) oversight was to neglect the overall algebraic h_r for $d = 3$. For d odd, numerically the h_r are of comparable size to the z_r and their interaction is essential to the final asymptotic form of the coefficients. This point contradicts the often assumed smallness of higher harmonics in Poisson summations, especially in C^∞ boundaries.

3. Fits to the late terms

The Weyl coefficients for arbitrary-dimensional balls can now be calculated symbolically from equations (23), (24) and (31), at least up to the limits of computer memory. A selection of the coefficients up to $d = 9$ is displayed in table 1. The results of this method agree with those of Bordag *et al* (1996a) and Levitin (1998), after appropriate scaling factors have been accounted for.

Having computed these figures, we follow BH and HT and examine the late-term behaviour. If our c_r do indeed fit the form of the conjecture (3), then we should be able to estimate values for the constants. Assuming the result holds, then

$$\tau(r) \equiv \frac{c_r c_{r-2}}{c_{r-1}^2} \sim \frac{r+\beta}{r+\beta-1} \Rightarrow \beta = \frac{\tau}{\tau-1} - r + \mathrm{O}\left(\frac{1}{r}\right).$$

Hence we use the function

$$B(r) = \frac{\tau}{\tau-1} - r \tag{32}$$

to find the value for β in each dimension. The dominant periodic orbit can then be deduced from considering the slope of the function

$$L(r) = \ln\left|\frac{c_r}{(r+\beta)!}\right| \sim r\ln l - \ln|\alpha| \tag{33}$$

plotted against r. Finally, estimates of α are obtained by plotting

$$A(r) = \frac{l^r c_r}{(r+\beta)!} \sim \alpha. \tag{34}$$

The graphs produced are shown in figures 1–3 and include further data from $d = 2$. These estimates are confirmed analytically below, and the superposed lines correspond to those predictions.

The results are surprising. The orbit dominating the late terms of the Weyl series of even-d balls, as in the circle, is found to be the shortest periodic path, with length 4. However, this

Table 1. The first ten coefficients c_r in the Weyl series for the Dirichlet sphere billiards in 3–9 dimensions.

	$d=3$	$d=4$	$d=5$	$d=6$
$r=1$	$\frac{1}{3}$	$-\frac{11\pi}{512}$	$-\frac{4}{945}$	$\frac{2159\pi}{1572\,864}$
$r=2$	$-\frac{1}{48}$	$-\frac{1}{180}$	$\frac{17}{11\,520}$	$\frac{1}{1512}$
$r=3$	$-\frac{1}{315}$	$-\frac{35\pi}{131\,072}$	$\frac{19}{45\,045}$	$\frac{1685\pi}{50\,331\,648}$
$r=4$	$-\frac{1}{960}$	$-\frac{29}{45\,045}$	$\frac{157}{967\,680}$	$\frac{571}{6235\,515}$
$r=5$	$-\frac{2}{3003}$	$-\frac{911\pi}{4194\,304}$	$\frac{1838}{14\,549\,535}$	$\frac{3512\,407\pi}{103\,079\,215\,104}$
$r=6$	$-\frac{47}{80\,640}$	$-\frac{13\,432}{14\,549\,535}$	$\frac{5}{39\,424}$	$\frac{4988}{31\,702\,671}$
$r=7$	$-\frac{3169}{4849\,845}$	$-\frac{4136\,575\pi}{8\,589\,934\,592}$	$\frac{3044}{18\,706\,545}$	$\frac{582\,966\,649\pi}{6597\,069\,766\,656}$
$r=8$	$-\frac{521}{591\,360}$	$-\frac{2911\,072}{1003\,917\,915}$	$\frac{593}{2396\,160}$	$\frac{7712\,057\,536}{13\,537\,833\,083\,775}$
$r=9$	$-\frac{198\,641}{143\,416\,845}$	$-\frac{1110\,131\,911\pi}{549\,755\,813\,888}$	$\frac{179\,214\,739}{410\,237\,366\,175}$	$\frac{238\,135\,098\,481\pi}{562\,949\,953\,421\,312}$
$r=10$	$-\frac{9521}{3843\,840}$	$-\frac{10\,032\,272\,896}{644\,658\,718\,275}$	$\frac{32\,815\,499}{37\,638\,881\,280}$	$\frac{247\,981\,260\,544}{71\,557\,117\,728\,525}$

	$d=7$	$d=8$	$d=9$
$r=1$	$\frac{349}{675\,675}$	$-\frac{260\,699\pi}{1509\,949\,440}$	$-\frac{11\,108}{138\,881\,925}$
$r=2$	$-\frac{367}{1935\,360}$	$-\frac{23}{226\,800}$	$\frac{27\,859}{928\,972\,800}$
$r=3$	$-\frac{1627}{24\,249\,225}$	$-\frac{16\,169\,407\pi}{3092\,376\,453\,120}$	$\frac{535\,004}{45\,176\,306\,175}$
$r=4$	$-\frac{23\,413}{851\,558\,400}$	$-\frac{1374\,409}{90\,352\,612\,350}$	$\frac{28\,291}{5635\,768\,320}$
$r=5$	$-\frac{51\,109}{2151\,252\,675}$	$-\frac{1181\,500\,627\pi}{197\,912\,092\,999\,680}$	$\frac{386\,811\,188}{83\,616\,027\,870\,375}$
$r=6$	$-\frac{188\,963}{7380\,172\,800}$	$-\frac{345\,483\,706}{11\,945\,146\,838\,625}$	$\frac{65\,826\,419}{12\,646\,664\,110\,080}$
$r=7$	$-\frac{367\,267}{10\,342\,118\,475}$	$-\frac{57\,386\,081\,761\pi}{3377\,699\,720\,527\,872}$	$\frac{8880\,814\,633}{1168\,766\,256\,232\,575}$
$r=8$	$-\frac{87\,132\,679}{1505\,555\,251\,200}$	$-\frac{858\,884\,026\,976}{7513\,497\,361\,495\,125}$	$\frac{655\,336\,327}{50\,586\,656\,440\,320}$
$r=9$	$-\frac{91\,432\,492\,744}{834\,833\,040\,166\,125}$	$-\frac{191\,235\,186\,442\,949\pi}{2161\,727\,821\,137\,838\,080}$	$\frac{113\,473\,229\,730\,632}{4415\,431\,949\,438\,635\,125}$
$r=10$	$-\frac{114\,193}{488\,816\,640}$	$-\frac{79\,950\,804\,764\,800}{105\,970\,366\,786\,527\,243}$	$\frac{2113\,097\,393\,809}{36\,843\,948\,107\,366\,400}$

is not the case in odd dimensions. Instead, it is the next-longest orbit with three bounces. In general,

$$l = \begin{cases} 4 & d \text{ even} \\ 3\sqrt{3} & d \text{ odd.} \end{cases} \tag{35}$$

We tackle the question of why this should be so below. The corresponding values for β are found to follow the pattern

$$\beta = \begin{cases} \dfrac{d-5}{2} & d \text{ even} \\ d - \frac{7}{2} & d \text{ odd.} \end{cases} \tag{36}$$

The estimates for α are consistent with the formulae (confirmed analytically below)

$$\alpha = \begin{cases} -\dfrac{2^{7/2-3d/2}\mathrm{i}^d}{\pi(d-2)!}\left(\dfrac{d-3}{2}\right)! & d \text{ even} \\ \dfrac{2^{1/2-d}3^{4-3d/2}\mathrm{i}^{3d+1}}{\sqrt{\pi}(d-2)!} & d \text{ odd.} \end{cases} \tag{37}$$

Due to the large number of coefficients that can be generated it is possible to test the extended conjecture (4) of HT by the use of a Neville table (Voros 1983 appendix B, HT appendix C). The estimates for the higher-order coefficients $\alpha_k^{(j)}$ for $d = 2$–4 are displayed in tables 2–4.

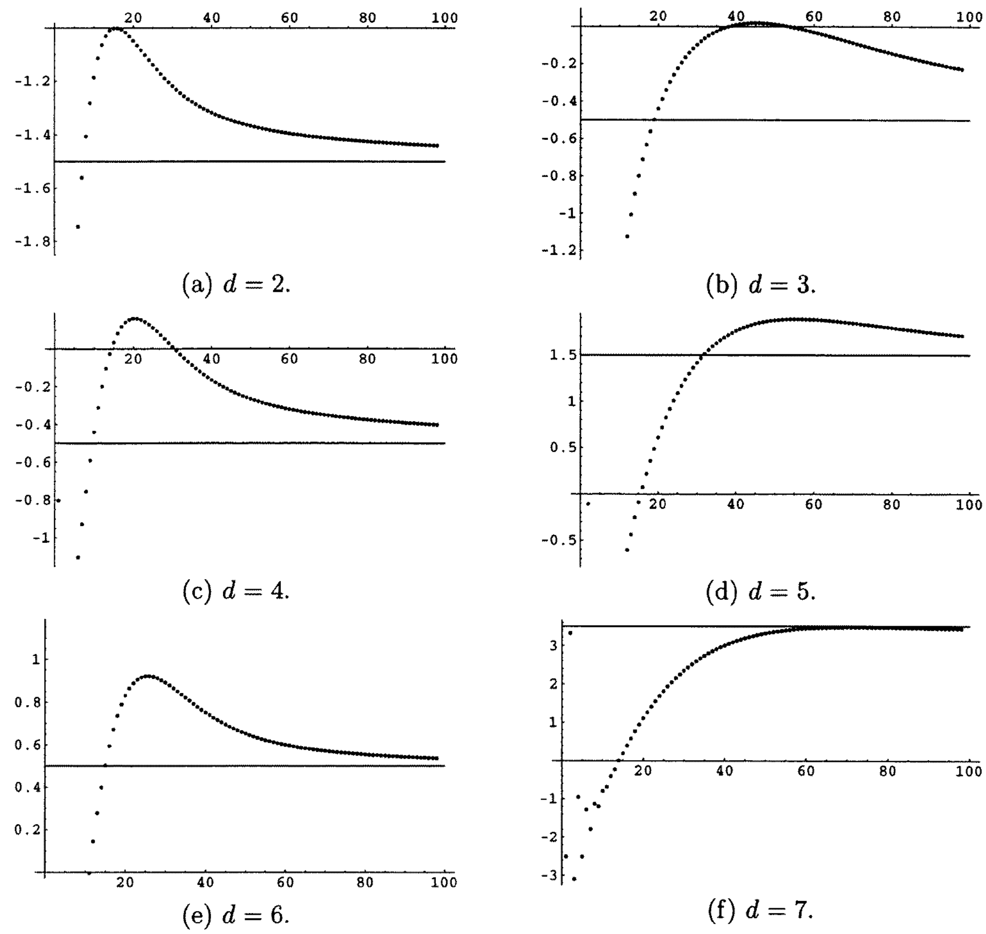

Figure 1. Plotting $B(r)$ versus r to estimate β in the conjecture (3) for the Dirichlet d-spheres with Dirichlet boundary conditions. Lines are drawn in for the predicted values (36).

Table 2. Neville table for the coefficients $\alpha_i^{(2)}$ in the expansion (4) for the circle billiard. The index p denotes the iteration of the Neville algorithm. The corresponding analytic predictions of section 4 are displayed at the bottom of the table.

p	α_0	α_1	α_2	α_3
1	0.776 325 0675	−0.530 899 5137	−0.178 062 5806	−0.372 128 2247
2	0.798 218 5879	−0.522 838 1627	−0.160 007 8416	−0.310 000 7395
3	0.797 846 9566	−0.523 723 6769	−0.163 030 1067	−0.319 078 0005
4	0.797 891 1460	−0.523 568 5682	−0.162 439 6126	−0.314 599 0688
5	0.797 882 5689	−0.523 610 8705	−0.162 808 5390	−0.318 381 6141
6	0.797 885 5371	−0.523 586 5667	−0.162 637 5765	
7	0.797 884 0772	−0.523 594 3822		
8	0.797 884 4386			
	0.797 884 5608	−0.523 611 7430	−0.161 680 7093	−0.341 854 8673

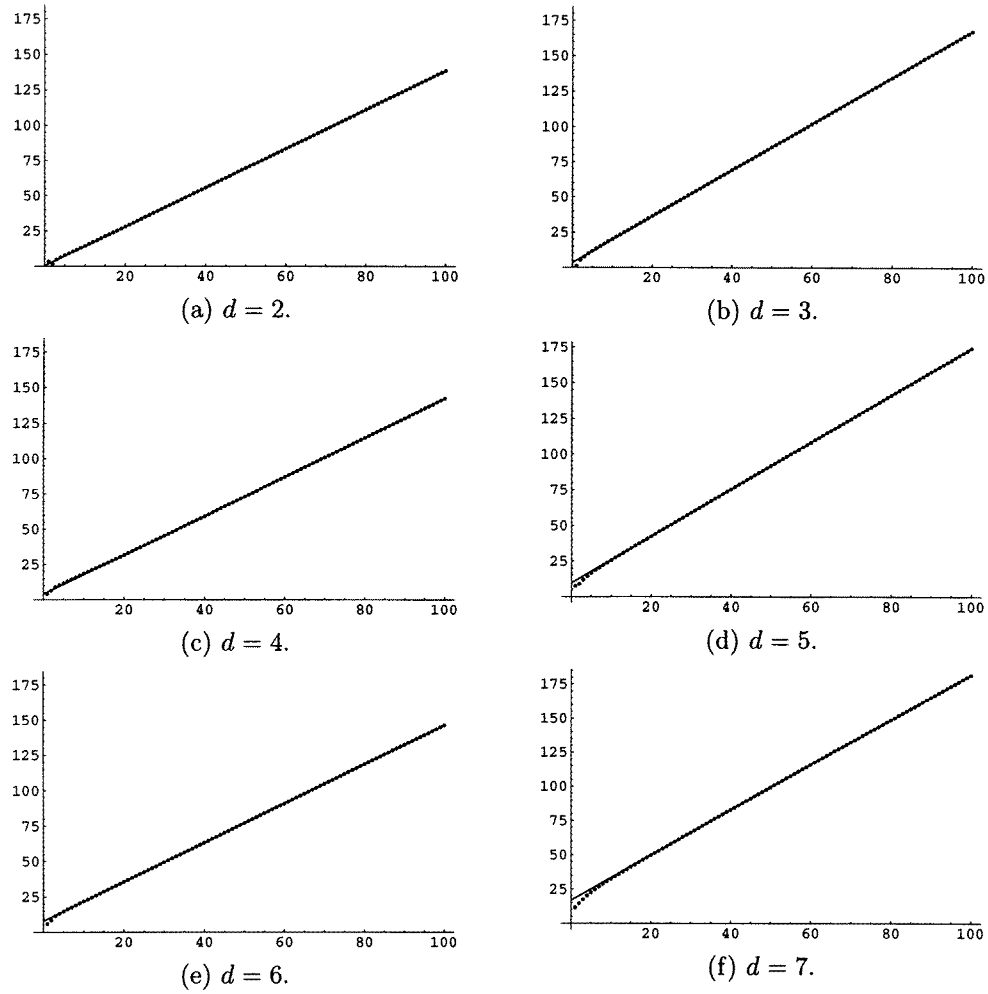

Figure 2. Plotting $L(r)$ versus r to estimate l in the conjecture (3) for the Dirichlet d-spheres. Lines of slope corresponding to the predicted lengths (35) are superimposed to highlight the agreement. Note the scales on the vertical axes are the same for comparison between even and odd dimensions.

Table 3. As above, but for the coefficients $\alpha_i^{(3)}$ in the expansion (4) for the sphere billiard.

p	α_0	α_1	α_2	α_3
1	−0.052 014 7449	0.109 712 8339	0.146 597 4253	6.768 357 7178
2	−0.057 997 7481	0.104 956 0081	0.378 559 3287	9.134 653 3181
3	−0.057 449 8351	0.110 366 8426	0.507 309 4084	
4	−0.057 350 8081	0.117 446 2610		
5	−0.057 744 2385			
	−0.057 582 3583	0.101 736 4380	0.058 154 8166	0.347 315 0204

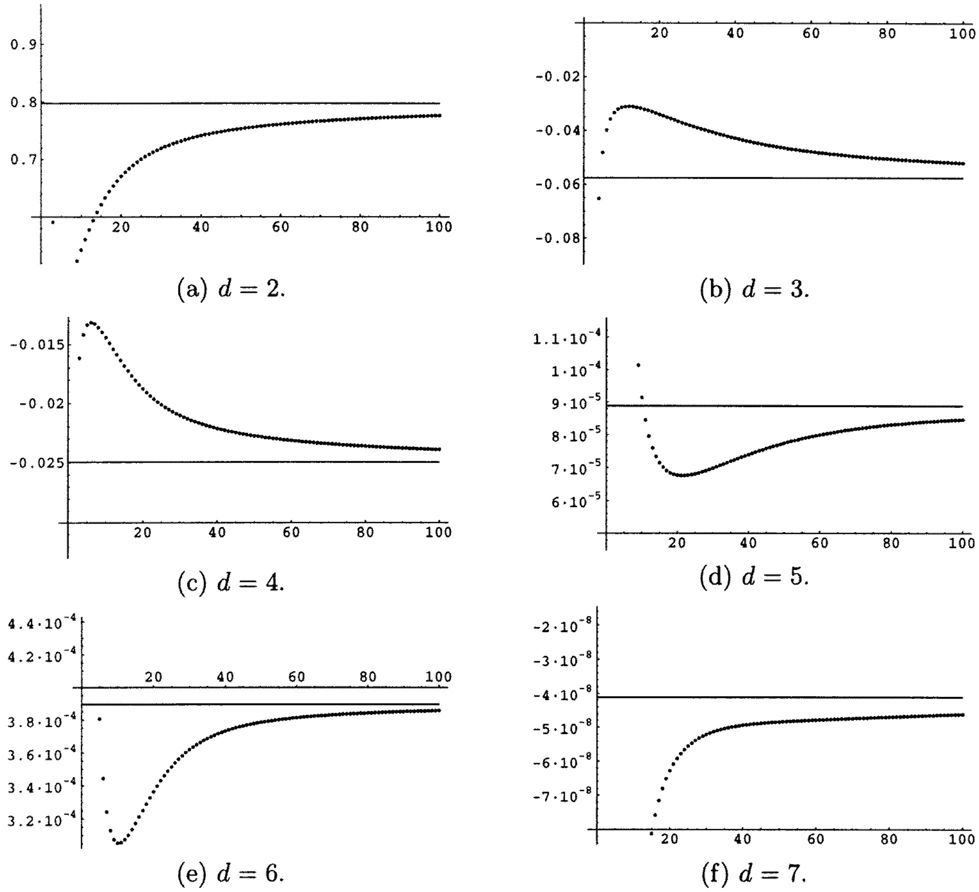

Figure 3. Plotting $A(r)$ versus r to estimate α in the conjecture (3) for the Dirichlet d-spheres with Dirichlet boundary conditions. The lines show the predicted values (37).

Table 4. As above, but for the coefficients $\alpha_i^{(2)}$ in the expansion (4) for the 4-sphere billiard.

p	α_0	α_1	α_2	α_3
1	−0.023 875 4236	0.026 334 4156	0.014 640 9996	0.052 701 9733
2	−0.024 961 7704	0.025 659 7937	0.012 709 4661	0.058 207 5182
3	−0.024 930 6420	0.025 742 2008	0.012 764 8315	0.064 891 9308
4	−0.024 934 7229	0.025 727 4912	0.012 478 8673	
5	−0.024 933 5530	0.025 739 8572		
6	−0.024 934 0936			
	−0.024 933 8925	0.025 713 0767	0.013 915 7418	0.025 778 1355

4. Analytic justification of HT conjecture

In BH, the analytical justifications for α, β, l when $d = 2$, were derived from a comparison of a Borel sum of the assumed approximate form (3) of c_r with a first-order asymptotic study of the exponentials generated by the Stokes phenomenon of $f_m(s)$ (15). In HT, the conjecture (4) was tested by expanding the high order c_r directly. Here we carry out a more complete analysis of the Stokes phenomenon affecting $f_m(s)$ which is responsible for generating the oscillatory periodic orbit corrections on the imaginary s (real energy) line. We compare the results with the Borel sum of the more detailed late-term conjecture (4) to identify the $\alpha_k^{(j)}$ ($j > 0$) and link the early terms of the periodic orbit expansion $c_r^{(j)}$ with the late Weyl c_r. To ensure that all the relevant exponentials have fully switched on we shall work with real energies $k = \mathrm{i}s$.

First, the assumed extended late-term form (4) for the c_r is inserted in the Weyl expansion and Borel-summed (Berry 1989). As the summed quantities undergo Stokes phenomena as s rotates to its imaginary axis (real k) (Berry and Howls 1991) the exponentials generated for each l_j take the form (cf BH (7))

$$g_{\mathrm{Borel}}^{(\mathrm{exp})}(s) \sim \mathrm{i}\pi(-\mathrm{i}kl_j)^{\beta_j+1}\mathrm{e}^{\mathrm{i}kl_j}\left(\alpha_0^{(j)} + \frac{\mathrm{i}\alpha_1^{(j)}}{k} - \frac{\alpha_2^{(j)}}{k^2} + \cdots\right). \tag{38}$$

This result is the basis for the comparison with the periodic orbit contributions which we now calculate.

From BH, all the relevant oscillatory periodic orbit contributions come from the ratio of Bessel functions in f_m alone. Denoting this by ϕ_m, for real k (imaginary s) we have the identity

$$\phi_m(k) = -\frac{J_m'(k)}{kJ_m(k)} = -\frac{1}{k}\frac{H_m^{1\prime}(k) + H_m^{2\prime}(k)}{H_m^1(k) + H_m^2(k)}. \tag{39}$$

We choose to take out the Hankel functions of the second kind (the reason is explained below) from the top and bottom and write this as

$$\phi_m(k) = -\frac{A_1}{k}\frac{1 + A_2\mathrm{e}^{2\mathrm{i}\psi}}{1 + A_3\mathrm{e}^{2\mathrm{i}\psi}} \tag{40}$$

$$A_1 = \frac{H_m^{2\prime}(k)}{H_m^2(k)} \qquad A_2 = \mathrm{e}^{-2\mathrm{i}\psi}\frac{H_m^{1\prime}(k)}{H_m^{2\prime}(k)} \qquad A_3 = \mathrm{e}^{-2\mathrm{i}\psi}\frac{H_m^1(k)}{H_m^2(k)} \tag{41}$$

where the function ψ is defined as

$$\psi(k, m) = \sqrt{k^2 - m^2} - \mathrm{i}m\ln\left(\frac{k}{m + \mathrm{i}\sqrt{k^2 - m^2}}\right) - \frac{\pi}{4}. \tag{42}$$

The large-order, large-argument Debye asymptotic expansions of the Hankel functions can be found in Abramowitz and Stegun (1972 section 9.3). The exponential behaviour in each of the Hankel functions is made explicit in the definitions (41) and the A_j are then asymptotically purely algebraic. The explicit factor $\mathrm{e}^{2\mathrm{i}\psi}$ in (40) is extracted since it will ultimately generate the required periodic orbit lengths. The analysis of BH corresponds to replacing the A_j by their lowest-order Debye approximations.

A formal binomial expansion of (40) generates

$$\phi_m(k) = -\frac{A_1}{k} - \frac{A_1}{k}\sum_{p=1}^{\infty}(-)^p A_3^p \mathrm{e}^{2\mathrm{i}\psi p}\left\{1 - \frac{A_2}{A_3}\right\}. \tag{43}$$

The contents of the braces multiplied by A_1 simplify by identifying the numerator of

$$A_1\left\{1 - \frac{A_2}{A_3}\right\} = \frac{H_m^1(k)H_m^{2\prime}(k) - H_m^{1\prime}(k)H_m^1(k)}{kH_m^1(k)H_m^2(k)} \tag{44}$$

as precisely the Wronskian

$$\mathcal{W}(H_m^1(k), H_m^2(k)) = -\frac{4\mathrm{i}}{\pi k}. \tag{45}$$

Ignoring the algebraic (Weyl) parts of (43), the exponential contribution from the leading-order part of the m-products (18)–(20) to the resolvent is found to be

$$L_\nu^{(\mathrm{exp})}(-\mathrm{i}k) \sim -\frac{\mathrm{i}}{(2\nu)!}\sum_{\mu=-\infty}^{\infty}(-)^{2\mu\nu}\sum_{p=1}^{\infty}(-)^p\int_0^\infty \mathrm{d}m\, m^{2\nu}\frac{\sqrt{k^2-m^2}}{k^2} \times\left(\sum_{j=0}^{\infty}\frac{(-)^j u_j(t)}{m^j}\right)^{p-1}\left(\sum_{j=0}^{\infty}\frac{u_j(t)}{m^j}\right)^{-(p+1)}\mathrm{e}^{2\mathrm{i}\psi p+2\pi\mathrm{i}m\mu} \tag{46}$$

where $t = m(m^2-k^2)^{-1/2}$ and the functions $u_j(t)$ are defined by the recurrence relation (Abramowitz and Stegun 1972 section 9.3),

$$u_{k+1}(t) = \frac{t^2(1-t^2)}{2}u_k'(t) + \frac{1}{8}\int_0^t \mathrm{d}\tau\,(1-5\tau^2)u_k(\tau) \qquad u_0 = 1. \tag{47}$$

Formally expanding the integrand to the first few (inverse) powers of m,

$$L_\nu^{(\mathrm{exp})}(-\mathrm{i}k) = -\frac{\mathrm{i}}{(2\nu)!}\sum_{\mu=-\infty}^{\infty}(-)^{2\mu\nu}\sum_{p=1}^{\infty}(-)^p\int_0^\infty \mathrm{d}m\, m^{2\nu}\frac{\sqrt{k^2-m^2}}{k^2} \times\left\{1-\frac{2pu_1(t)}{m}+\frac{(1+2p^2)u_1^2(t)-2u_2(t)}{m^2} -\frac{2p}{3}\frac{2(p^2+2)u_1^3(t)-9u_1(t)u_2(t)+3u_3(t)}{m^3}+\cdots\right\}\mathrm{e}^{2\mathrm{i}\psi p+2\pi\mathrm{i}m\mu}. \tag{48}$$

These integrals are evaluated by the method of steepest descent (Dingle 1973). The saddles m_0 lie at

$$m_0 = k\cos\frac{\pi\mu}{p}. \tag{49}$$

Thus the $p = 2\mu$ saddles coincide with the integration endpoint $m = 0$ and so have to be treated separately from the rest. This segregation does not occur in two dimensions because the integrals are all doubly infinite (BH).

4.1. Odd dimensions: $p > 2\mu$ contributions

First we concentrate on odd dimensions where the late terms of the Weyl series are dominated by the $p > 2\mu$ orbits. Contributions to each order of k arise from several sources and care is needed to include them all. Take the example of the 7-sphere. In terms of the L_ν, the contributions to the resolvent arise from (cf equation (22))

$$W_7 = L_{5/2}(s) - \tfrac{1}{1920}W_3 - \tfrac{1}{8}W_5 \tag{50}$$

$$= L_{5/2}(s) - \tfrac{1}{8}L_{3/2}(s) + \tfrac{3}{640}L_{1/2}(s). \tag{51}$$

Every component of the braces in each L_ν (cf equation (48)) generates a series in increasing powers of k^{-1} with the same exponential periodic orbit prefactor: if the series expansion corresponding to the first term in the braces of (48) starts at $\mathrm{O}(k^{-r})$, that of the second will start at $\mathrm{O}(k^{-r-1})$ and so on. In addition, if the expansion for L_ν starts at $\mathrm{O}(k^{-n})$, that for $L_{\nu-1}$ will start at $\mathrm{O}(k^{-n-2})$.

Expanding the integrals up to the relevant higher orders in k using, for example, Dingle (1973 pp 135, 118), we find (for $p \geqslant 3$)

$$L_\nu^{(\mathrm{exp})}(k) = -\frac{1}{(2\nu)!}\sum_{\mu=-\infty}^{\infty}\sum_{p=3}^{\infty}(-)^{2\mu\nu}\mathrm{i}^{p+3/2}\mathrm{e}^{\mathrm{i}kl}y^{3/2}x^{2\nu}k^{2\nu-1/2}\sqrt{\frac{\pi}{p}}\sum_{r=0}^{\infty}T_r \tag{52}$$

$$T_0 = 1 \tag{53}$$

$$T_1 = -\frac{\mathrm{i}(24\nu(1-2\nu) + 3x^2(5+4p^2+8\nu+32\nu^2) - x^4(11-8p^2+48\nu+48\nu^2))}{48kpy^2x^2} \tag{54}$$

$$\begin{aligned} T_2 = -\frac{1}{4608k^2p^2y^6x^4}&[48\nu x^2(1-2\nu)(107+12p^2-152\nu+96\nu^2)\\ &-576\nu(3-11\nu+12\nu^2-4\nu^3) - 3x^4(61-48p^2+1920\nu-5344\nu^2\\ &+5376\nu^3-4608\nu^4+8p^2(61+168\nu-64\nu^2)) - 6x^6(63-32p^4-408\nu\\ &+784\nu^2-384\nu^3+1536\nu^4+4p^2(173-144\nu-16\nu^2))\\ &-x^8(1+64p^4-96\nu-96\nu^2+1536\nu^3+2304\nu^4+16p^2(1-48\nu^2))] \end{aligned} \tag{55}$$

$$\begin{aligned} T_3 = -\frac{\mathrm{i}}{3317\,760k^3p^3y^9x^6}&[8640\nu x^2(367+12p^2-328\nu+96\nu^2)(3-11\nu+12\nu^2-4\nu^3)\\ &-69\,120\nu(30-137\nu+225\nu^2-170\nu^3+60\nu^4-8\nu^5)\\ &+1080\nu x^4(2\nu-1)(15\,987+48p^4-40\,936\nu+39\,600\nu^2-18\,560\nu^3\\ &+3840\nu^4+8p^2(111-320\nu+80\nu^2))\\ &-45x^6(3351+192p^6-328\,840\nu+1589\,760\nu^2-2634\,752\nu^3+2204\,160\nu^4\\ &-1044\,480\nu^5+245\,760\nu^6+48p^4(63-360\nu+32\nu^2)\\ &+4p^2(11\,793+35\,584\nu+76\,128\nu^2-60\,160\nu^3+7680\nu^4))\\ &+9x^8(123\,843+1920p^6+812\,360\nu-3668\,000\nu^2+5821\,440\nu^3\\ &-5107\,200\nu^4+2918\,400\nu^5-921\,600\nu^6+16p^4(15\,927+1040\nu+880\nu^2)\\ &+80p^2(22\,683+5140\nu-24\,712\nu^2+10\,560\nu^3))\\ &-9x^{10}(37\,489+1280p^6-261\,880\nu+698\,560\nu^2-956\,160\nu^3+902\,400\nu^4\\ &-768\,000\nu^5+368\,640\nu^6+128p^4(837+460\nu-40\nu^2)\\ &+20p^2(30\,957+59\,744\nu+41\,696\nu^2-7680\nu^3-3840\nu^4))\\ &+x^{12}(1183-2560p^6-16\,560\nu-16\,560\nu^2+253\,440\nu^3-172\,800\nu^4\\ &-552\,960\nu^5+552\,960\nu^6+192p^4(151-240\nu+240\nu^2)\\ &+120p^2(23-1056\nu^2+3072\nu^3-2304\nu^4))] \end{aligned} \tag{56}$$

where

$$y = \sin\frac{\pi\mu}{p} \qquad l = 2py \qquad x = \cos\frac{\pi\mu}{p} = \sqrt{1-\left(\frac{l}{2p}\right)^2}. \tag{57}$$

This gives the leading-order term and just the first three corrections! However this expansion, when used in conjunction with expressions of the form (50) is actually sufficient to generate the first four terms $c_r^{(j)}$ ($r = 0$–3), corresponding to the orbit of choice ($p \geqslant 3 \geqslant 2\mu$) for any sphere with $d \geqslant 3$ (regardless of dimensional parity).

Focus now on the contributions (52)–(56) specific to the 3-bounce orbit, of interest in odd dimensions. They are found by setting $p = 3$, $\mu = 1$ and $l = 3\sqrt{3}$. The values of y and x follow and

$$L_\nu^{(\mathrm{exp})}(k)|_{(3,1)} = -\frac{(-)^{2\nu}\mathrm{e}^{\mathrm{i}kl}k^{2\nu-1/2}\sqrt{\pi\mathrm{i}\sqrt{3}}}{(2\nu)!2^{2\nu+3/2}}\sum_{r=0}^{\infty}T_r$$

$$T_0 = 1 \tag{58}$$

$$T_1 = -\frac{\mathrm{i}}{3^3 2^3 k\sqrt{3}}(553+432\nu-432\nu^2) \tag{59}$$

$$T_2 = \frac{1}{3^7 2^7 k^2 \sqrt{3}} (110\,567 + 320\,544\nu - 299\,808\nu^2 + 787\,968\nu^3 - 186\,624\nu^4) \tag{60}$$

$$T_3 = \frac{\mathrm{i}}{3^{11} 2^{21} k^3 \sqrt{3}} (1686\,372\,121 + 584\,943\,120\nu - 1353\,896\,208\nu^2 + 2947\,788\,288\nu^3 - 2189\,659\,392\nu^4 + 779\,341\,824\nu^5 - 80\,621\,568\nu^6) \tag{61}$$

for this family of orbits.

Observing the overall algebraic prefactor is proportional to $k^{2\nu-1/2} = k^{d-2}$, from these coefficients and by comparison with (38) we see that the value of β can be confirmed as $d - \frac{7}{2}$. This result can be checked by comparison with the work of Balian and Bloch (1972). They predict (HT 56) an algebraic prefactor of $k^{q_0/2-1}$ for a contribution to the exponential part of the resolvent of a C^∞ billiard from a periodic orbit of degeneracy q_0. The value of β here gives $q_0 = 2d - 3$, the correct degeneracy of a 3-bounce orbit in a d-ball.

Only the first term in the braces of the specific L_ν in (48) is required to find the leading-order late-term Weyl behaviour in any odd dimension $d > 2$. By comparison with (38), it is found to be (cf (37))

$$\alpha_0^{(3)} = \frac{2^{1/2-d} 3^{4-3d/2} \mathrm{i}^{3d+1}}{\sqrt{\pi}(d-2)!} \tag{62}$$

and $l = 3\sqrt{3}$. This result agrees with the numerical prediction for odd dimensions in section 3. Further comparison with (38) reveals the next three corrections. In three dimensions, these are

$$\alpha_0^{(3)} = -\frac{1}{4\sqrt{6\pi}} \qquad \alpha_1^{(3)} = \frac{661}{2592\sqrt{2\pi}}$$
$$\alpha_2^{(3)} = \frac{282\,719}{1119\,744\sqrt{6\pi}} \qquad \alpha_3^{(3)} = \frac{1895\,084\,173}{2176\,782\,336\sqrt{2\pi}}. \tag{63}$$

Knowing the coefficients in the expansion (22), analytic predictions for the $\alpha_r^{(j)}$, $r > 0$ could be found in any odd dimension.

4.2. Even dimensions: $p = 2\mu$ contributions

Now we examine the coefficients for the 2-bounce (or more generally the $p = 2\mu$) orbit expansion. Since the corresponding integrals L_ν contain quadratic endpoints the series expansion should be in powers of $k^{-1/2}$. However, due to the nature of the integrand, half the coefficients are identically zero, recovering an integer-order expansion.

Using Dingle (1973 p 118), we break down the (2, 1) contribution as follows:

$$L_\nu^{(\mathrm{exp})}(k)|_{(2,1)} = -\frac{(-)^{2\nu} \mathrm{i}^{\nu+1/2} k^{\nu-1/2} \mathrm{e}^{\mathrm{i}kl}}{2^{d/2+1/2}(2\nu)!} \left(\frac{d-3}{2}\right)! + \mathrm{O}(k^{\nu-3/2}). \tag{64}$$

Again, the factor of β in even-d balls can be checked. Comparison with (38) generates $\beta = \frac{1}{2}(d-5)$. In turn this gives a value of $q_0 = d-1$, the correct (lower rotational) degeneracy of a 2-bounce orbit in a d-ball.

In even dimensions (where the $p = 2\mu$ orbit dominates the Weyl series), $\alpha = \alpha_0^{(2)}$ is given by the first term of the expansion (64) (cf (38)) and yields

$$\alpha_0^{(2)} = -\frac{2^{5/2-3d/2} \mathrm{i}^{3d}}{\pi(d-2)!} \left(\frac{d-3}{2}\right)! \tag{65}$$

which again agrees with the numerical prediction in section 3. The higher-order corrections can then be found relative to this leading term by summing the the relevant terms and comparison with (38). In four dimensions,

$$\alpha_0^{(2)} = -\frac{1}{16\sqrt{2\pi}} \qquad \alpha_1^{(2)} = \frac{33}{512\sqrt{2\pi}}$$
$$\alpha_2^{(2)} = \frac{1143}{32\,768\sqrt{2\pi}} \qquad \alpha_3^{(2)} = \frac{67\,755}{1048\,576\sqrt{2\pi}}. \tag{66}$$

In two dimensions, we have

$$\alpha_0^{(2)} = \frac{2}{\sqrt{2\pi}} \qquad \alpha_1^{(2)} = -\frac{21}{16\sqrt{2\pi}}$$
$$\alpha_2^{(2)} = -\frac{415}{1024\sqrt{2\pi}} \qquad \alpha_3^{(2)} = -\frac{28\,079}{32\,768\sqrt{2\pi}}. \tag{67}$$

The results (63), (66) and (67) are the exact values quoted in the Neville tables. The consistency of the exact and estimated values is dictated by the proximity of the next-longest to the dominant orbit (HT): this difference is smaller for d odd than for d even by a factor of 2.6, so the agreement is better in even dimensions. Note that since these $\alpha_k^{(j)}$ arise explicitly from the periodic orbit corrections of $g(s)$ and are consistent with numerical asymptotics of the c_r, we may conclude that, in this case, $\alpha_k^{(j)} = c_k^{(j)}$, thus confirming the resurgence link between the periodic orbit and Weyl terms.

5. Explanation of vanishing 2-bounce orbit in odd dimensions

We aim to understand why the 2-bounce orbit does not appear in the Weyl series for odd-d spheres, even though it has to appear in the periodic orbit. There are at least two approaches to understanding this phenomenon.

First, a more detailed analysis of the individual late-term behaviour of the Debye expansions (Abramowitz and Stegun 1972 section 9.3) is a model exercise for hyperasymptotic methods (Berry and Howls 1991) using contour integral representations of the Bessel functions in the ratio of (15). However, the ratio and additional algebraic contributions from the higher harmonics in (31) dramatically complicate and obfuscate the analysis†. Indeed a more complete explanation of the late terms both of the Weyl series and the periodic orbit corrections (section 6) would come with the rigorous application of the theory of resurgence as developed in Ecalle (1981, 1984) and Voros (1983, 1992). An attempt towards this forms the bulk of Howls and Trasler (1999).

For now, we exploit an understanding of the structure of asymptotic expansions given by Berry and Howls (1991), Howls (1992), Olde Daalhuis (1997), Howls (1997). Balian and Bloch (1972) reduced the analytic behaviour of spectral functions to a determination of the singularity structure of an associated complex length Λ plane. In the notation of Voros (1983), Olde Daalhuis (1997), Howls (1997) this is the Borel plane, with Λ the Laplace-dual variable of the energy s. The resolvent can be represented formally in (for example) $d = 3$ as the following integral

$$g(s) = \frac{s|\Omega|}{4\pi} - \frac{1}{2\pi}\sum_{p=1}^{\infty}\int_{C_p} \mathrm{d}\Lambda\, G_p(\Lambda)\mathrm{e}^{-s\Lambda} \tag{68}$$

† It is easy to show that

$$y_m(s) = \frac{I_m'(s)}{I_m(s)} \qquad \text{satisfies} \qquad y_m'(s) = 1 + \frac{m^2}{s^2} - y_m^2(s) - \frac{y_m(s)}{s}$$

a nonlinear Ricatti equation (Bender and Orszag 1978 section 1.6). The index m must be scaled with s, so a more elegant and precise late-term analysis would involve hyperasymptotic methods for nonlinear partial differential equations which is in the process of being developed (Olde Daalhuis 1998).

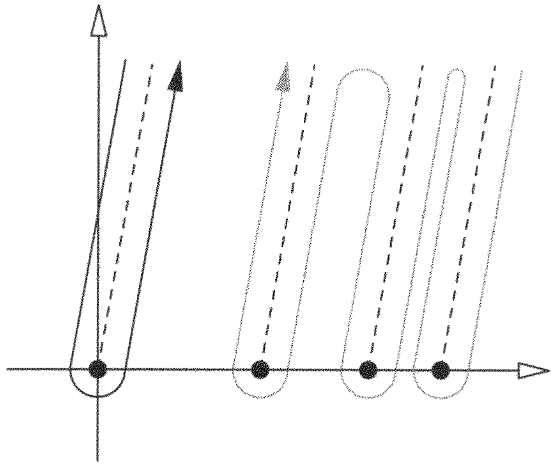

Figure 4. The contour (black line) around the Weyl singularity at the origin of the *length plane* is deformed outwards, giving the Weyl series instead as a sum over the *doubly-infinite* contours (grey lines) around adjacent singularities. Broken lines denote branch cuts.

where G_p is the 'path generating function' for the p-bounce orbits and C_p is an associated the contour of integration (Balian and Bloch 1972). Each p-integral generates a contribution to the Weyl series from an integral about $\Lambda = 0$. Periodic orbits contribute singularities of known type to each G_p elsewhere in the Λ-plane. Hyperasymptotic techniques (Berry and Howls 1991, Howls 1992, Howls 1997) show that the contributions to late terms of integral expansions arise from *doubly-infinite* contours over self-similar integrands around such distant adjacent singularities. The contour encircling the distant singularity corresponding to the $p = 2$, $\mu = 1$ orbit which is responsible for generating the contribution to the late terms of the Weyl series is shown in figure 4 (note that this analysis is only accurate up to the whispering-gallery mode contribution at $O(e^{-\mathrm{Re}\,(2\pi s)})$, which is sufficient for our purposes). This technique is an evolution of the work of Ecalle (1981, 1984), Voros (1983, 1992) and is explained further in Trasler (1998) and Howls and Trasler (1999), drawing on the work of Delabaere and Howls (1999).

In the original m and s variables of this paper, the contribution of the $p = 2$, $\mu = 1$ orbit to the c_r in $d = 2(\nu + 1)$ dimensions is composed of the integrand of the periodic orbit correction (46) together with other integrands effectively differing only by powers of m^2, (18)–(20). However, the m-image of the contour is over a *doubly*-infinite range. This is the crucial point because, if and only if $p = 2\mu$ the integrand is then either even or odd, the parity being determined by the power of the algebraic m-dependence. Consequently in odd dimensions (ν half-integer) the integral over the doubly-infinite range identically vanishes. Thus there is no contribution to the late terms of the Weyl series from $p = 2\mu$ orbits in odd dimensions. In even dimensions with $p = 2\mu$ there is no such cancellation and so the shortest, $p = 2$, $\mu = 1$ orbit dominates the c_r. Note that over a half-range in any dimension the integral (46) does not vanish, and so contributes the correct expansion for $p = 2\mu$ to the periodic orbit corrections.

6. Late terms of periodic orbit expansions

We can use the results of section 4 to make further conjectures concerning late terms of the high-energy expansions, but now on the periodic orbit corrections $c_r^{(j)}$. A study of the late

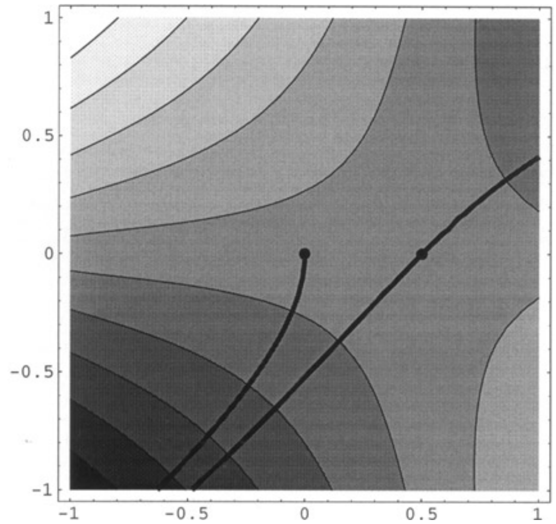

Figure 5. The contours of the real part of the exponent in (48) with $p = 3$, $\mu = 1$ and the contour of integration (full curve) superimposed.

terms in the expansion of integrals such as (46) is required. Without loss of generality we pick the $p = 3$, $\mu = 1$ orbit as an example. This orbit dominates the late terms of the Weyl series in odd dimensions, but do the early terms of the Weyl series (c_r) somehow dominate the late terms of the periodic orbit expansions ($c_r^{(j)}$)? This question can be answered by consideration of the topography of the exponent in (46), together with the singularity structure of the integral.

Using the scaling $kv = m$, in figure 5 we plot the real and imaginary parts of the exponent

$$\mathrm{i}(h(v) - h|_{\text{saddle}}) \qquad \text{with} \quad h(v) = 2p\psi(v) + 2\pi k v\mu, \quad (p, \mu) = (3, 1)$$

in the v-plane to obtain the contours of steepest descent. The dark-grey shading corresponds to 'valleys', light to 'hills'.

The steepest contour runs from the origin into the valley in the southwest, returning heading northeast through the saddle at $v = \frac{1}{2}$ and into the valley beyond. This picture is repeated for general $p > 2$.

In the hyperspheres, each (p, μ) integral has only one saddle. When $p \geqslant 3$, the origin is a linear endpoint and so it will not contribute to any re-expansion about the saddle at $v \neq 0$ (Howls 1992). (It will make a small correction at the first stage, from the contour leaving the origin in figure 5 and to compensate in (48) for our extending the integral out to infinity in both directions. However, this is unimportant for what is being discussed here.)

Thus from the work of Howls (1997) and the earlier discussion of section 4 the only other causes of the ultimate divergence of the large k expansion are the branch points of the square root in the algebraic prefactor of (46), at $v = \pm 1$. (Repeat orbits and hence images of the saddle at $v = \frac{1}{2}$ are not involved because the integrals are segregated with respect to both p and μ (cf section 5). Therefore the individual (p, μ) integrals do not 'see' each other directly.)

Each of the integrals under study are similar to those considered by Berry and Howls (1991) and Howls (1992, 1997). Thus from above we can deduce that each of their contributions to the late terms $c_r^{(j)}$ will diverge as a factorial over a power of the difference in the exponent between the saddlepoint and the branch point. Evaluating the exponent at the branch point we obtain

$$2\mathrm{i}\psi(k,m)p + 2\pi\mathrm{i}m\mu = -\frac{\mathrm{i}\pi p}{2} + \mathrm{i}k \times \begin{cases} 2\pi\mu & m=+k \\ 2\pi(p-\mu) & m=-k. \end{cases} \tag{69}$$

On removing the phase factor which is independent of k we are left with multiples of the length of whispering-gallery modes in the d-balls. For the orbit under consideration to be physical, we must have $p \geqslant 2\mu$: it then has length l given by (57). Under this condition, the second choice in (69) is either coincident with or lies to the right of the first (on the positive-real axis) in the length plane, which in turn is to the right of the image of the (sole) saddle. Thus the first choice is closest to the orbit in question: the dominating factor in the late-term behaviour is the distance of the orbit length from the whispering-gallery mode with the same number μ of turns about the origin.

Consequently we can adapt the general conjecture (3) on the oscillatory part of the level density for the hyperspheres. Given the form (2), we predict that

$$c_r^{(j)} \sim \frac{\alpha(r+\beta)!}{(w_j - l_j)^r} \qquad r \to \infty. \tag{70}$$

Here w_j is the length of the shortest (i.e. of all integer multiples) whispering-gallery mode longer than l_j. In the (resurgent) corrections of the second iteration, the late terms behave as a factorial over the *difference* between the lengths of orbit and the next-longest whispering-gallery mode. That the $c_r^{(j)}$ in billiard systems depend on the difference between orbit lengths is consistent with the findings of Boasman and Keating (1995) who considered quantum maps†.

7. Discussion and conclusions

We have demonstrated that the extended conjecture (4) suggested in HT for high orders of Weyl series for 2D billiards can hold in higher dimensions. However, even the most geometrically 'simple' billiards can spring surprises.

In HT the high c_r were expanded directly to observe the predicted form, but the higher-order corrections in (4) were not linked explicitly to the low orders of the periodic orbit corrections $c_{j,r}^{(p)}$. Here we have derived the low orders of the periodic orbit corrections $c_{j,r}^{(p)}$ and hence demonstrated the resurgence relation directly.

Using an extension of Stewartson and Waechter's (1971) and Kennedy's (1979) methods, we have been able to conjecture the leading asymptotic behaviour of the c_r in any dimension. This partially refutes one of the criticisms which were levelled by Levitin (1998) who claimed that knowledge of these methods in a particular dimension did not necessarily generate information about the c_r in another. It is clear that only the parity of the dimension is important, and knowledge of the first few dimensions is sufficient to conjecture the dominant asymptotic behaviour of the rest.

The absence of influence on the c_r by the shortest orbit in odd-d balls at the asymptotic level of (4) is for a different reason to the cases considered by BH. There the billiards where the shortest orbit failed to dominate were concave, and an explanation was provided which depended on the topography of the chord-length surface. The argument was reminiscent of the *adjacency* rules concerning contributions from distant critical points to integrals outlined by Berry and Howls (1991), Howls (1992) and Howls (1997). Here the vanishing of the 2-bounce in odd-dimensional Weyl terms is almost 'accidental', relying on a precise, sensitive,

† These results are also relevant to the circle billiard since the only difference due to dimensionality is the absence of a finite endpoint on the contour. The singularity structure which generates the whispering-gallery contributions is common to all dimensions.

symmetry property of an integrand. For that reason we suspect that a general slight perturbation in the boundary of the odd-d ball would lead to billiards where the shortest orbit would again dominate the c_r. This idea is examined further for 3D bodies of revolution in Howls and Trasler (1999).

It should be recalled that as a result of a different degeneracy the 2-bounce orbits have smaller amplitudes in the periodic orbit correction terms (for the spectral density) than the regular polygons in $d \geqslant 3$ (Balian and Bloch (1972) cf section 4). Thus the influence of the 2-bounce orbit in the fluctuations about the Weyl series is diminished in higher dimensions. It is interesting to contrast this behaviour with the role this orbit has in the Weyl series itself: the 2-bounce orbit either dominates or is completely absent, depending only on dimensional parity. An explanation of this in terms of orthogonal groups with even or odd Cartan classification also seems desireable, but is beyond the scope of this initial, more exploratory article.

The argument explaining the absence of influence of the 2-bounce on the c_r, but the contribution of the same orbit to the periodic orbit corrections is a subtle application of hyperasymptotic techniques. Such ideas underpin the ideas of BH and HT, but necessarily involve representing the spectral functions as Borel transforms. The work of Balian and Bloch (1972) coupled with the resurgence approach of Ecalle (1981, 1984), Voros (1983, 1992) and the results of Howls (1997), Delabaere and Howls (1999) should provide a more complete analytic explanation of the conjecture (4) in more general billiard systems. This approach will be outlined elsewhere (Howls and Trasler 1999).

This paper has made great use of saddlepoint techniques and Stokes phenomena. At times it has led to somewhat untidy, complicated and formal calculations. However, because of the simplicity of the boundary geometry we hope that others might provide a rigorous proof (or otherwise) of our conjectures. The method of Bordag *et al* (1996a, b, c) potentially lends itself to a more elegant examination of the asymptotic properties of the c_r, but the periodic orbits and consequently any resurgence relations have yet to be teased from that formalism.

Finally other types of boundary conditions on the balls should be examined. Neumann conditions and annuli are considered in Howls and Trasler (1999).

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